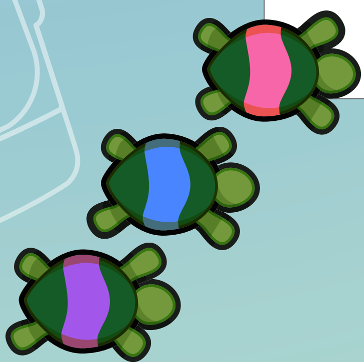
Unit 3

Proportional and Linear Relationships

The slope of a line tells us about its steepness, but did you know it can reveal even more? A simple line on a graph holds *endless* information. A line can tell us what's faster: a tortoise or a hare. It can help us sort coins in a piggy bank. It can even help us determine how many cups a water cooler can fill. Let's explore proportional and linear relationships, and learn all about what a line can represent!

Essential Questions

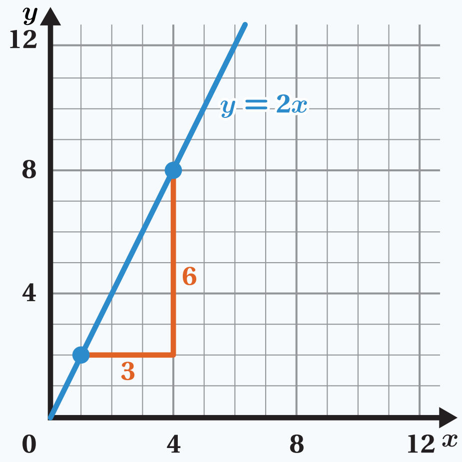
- What can proportional relationships and slope teach you about linear relationships?
- How do equations, tables, and graphs of linear relationships connect to one another?
- What does it mean for an ordered pair to be a solution to a linear equation?



Summary | Lesson 1

A line that passes through the origin, $(0, 0)$, represents a *proportional relationship*. The *slope* of the line represents a *unit rate* for this relationship.

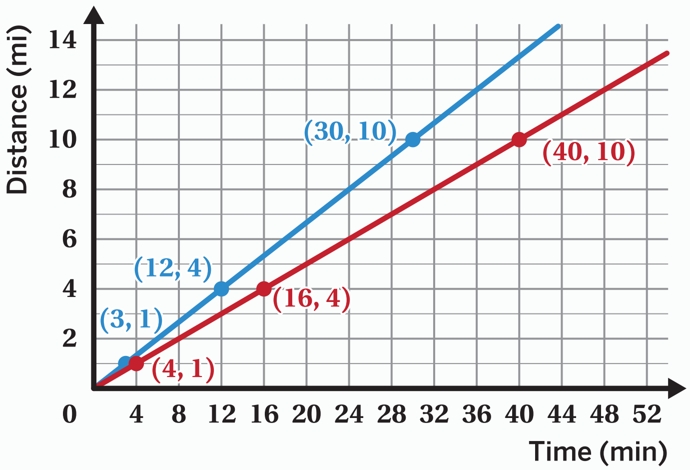
Here is a graph of the equation $y = 2x$. The slope of the line is $\frac{6}{3}$, or 2.



Try This

This graph shows the distance that Jasmine and Sothy travel on a long bike ride.

Jasmine rides 4 miles every 16 minutes, and Sothy rides 4 miles every 12 minutes.



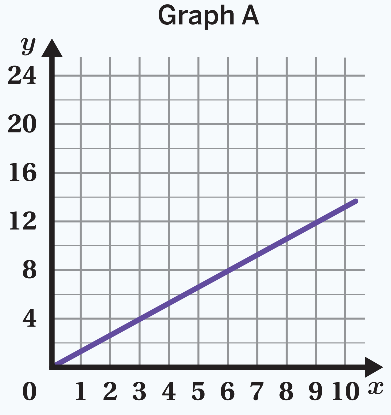
a Do these lines represent proportional relationships? Explain your thinking.

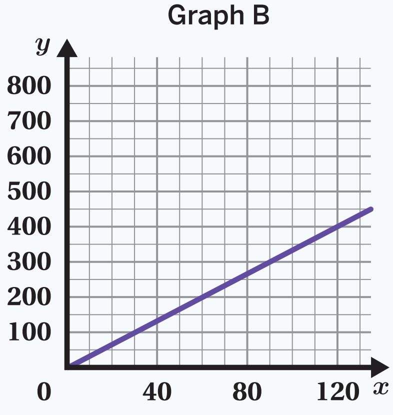
b What is the speed of each rider as a unit rate?

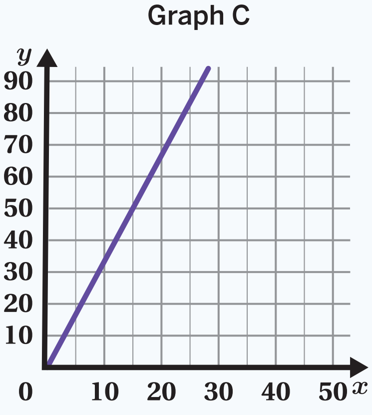
Summary | Lesson 2

We can represent a proportional relationship using the equation $y = mx$, where m represents both a unit rate and the slope of the line.

When we represent these relationships on axes with different scales, we can use slope to compare these graphs. For example, you can compare Graphs A, B, and C using their slopes.







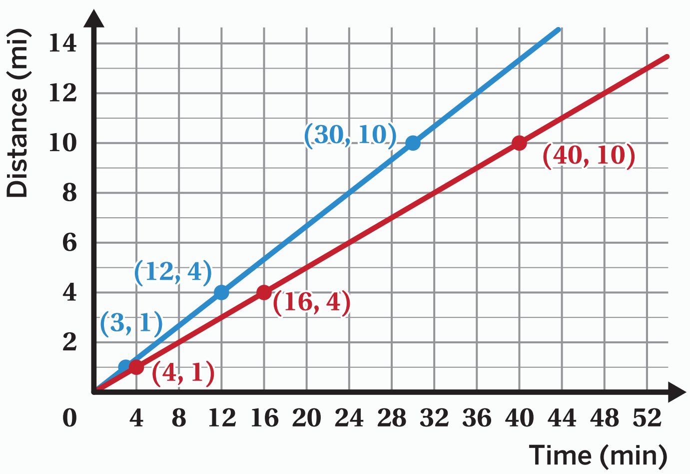
- Graph A slope: $\frac{12}{9} = \frac{4}{3}$
- Graph B slope: $\frac{100}{30} = \frac{10}{3}$
- Graph C slope: $\frac{50}{15} = \frac{10}{3}$

You can see that the slopes for Graph B and Graph C are equivalent. This means they have the same proportional relationship, even though the lines may look like they don't have the same steepness.

Try This

This graph shows the distance that Jasmine and Sothy travel on a long bike ride.

Jasmine rides 4 miles every 16 minutes, and Sothy rides 4 miles every 12 minutes.



a Determine the slope of each line.

b Write an equation to represent each line.

Summary | Lesson 3

Using different representations is helpful when comparing proportional relationships, such as equations, tables, and graphs.

You can represent proportional relationships with the equation $y = mx$, where m is the slope of a line and also represents a unit rate for the situation. You can identify the slope or unit rate using all of the different representations, or by using slope triangles within a graph.

Try This

As of 2024, the federal minimum wage is $7.25 per hour.

Create an equation, table, and graph to represent the relationship between hours worked and money earned for a person making minimum wage.

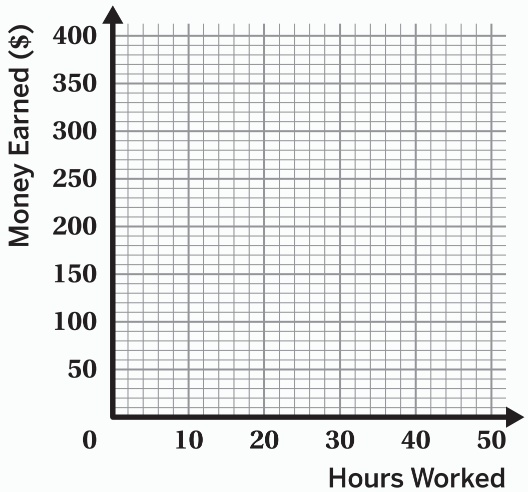
Table

Hours Worked	Money Earned ($)

Equation

____

Graph

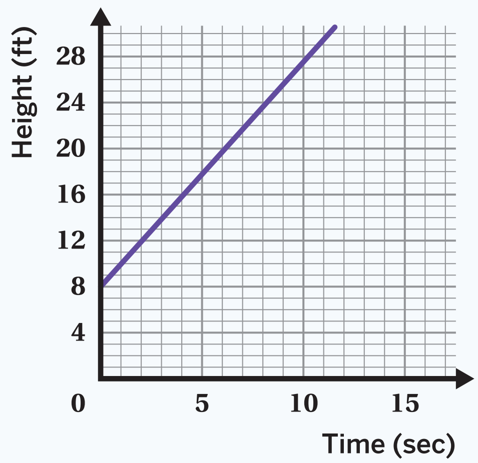


Linear relationships are graphs that are lines. Some linear relationships are proportional relationships and some are not.

For example, this graph represents a flag's height, in feet, over time.

- The line starts at $(0, 8)$, which means the flag is at first 8 feet off the ground.
- The slope is 2, which represents the number of feet the flag rises each second.
- The equation $y = 8 + 2x$ represents the height of the flag y after x seconds.

This relationship is linear, but it's non-proportional because the line does not pass through the origin.



Try This

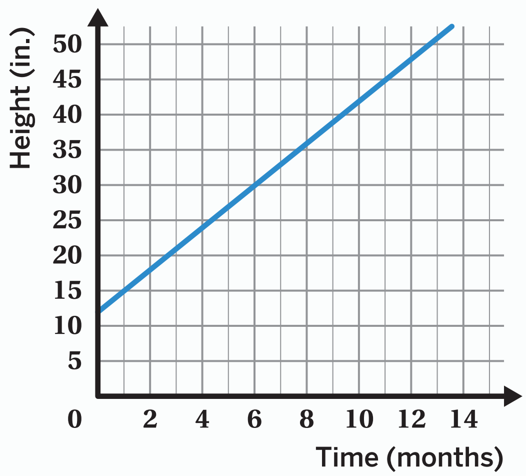
The graph shows the height, in inches, of a bamboo plant each month after it was planted.

The equation for this line is $y = 3x + 12$.

a What is the slope of this line? What does this value mean in context?

b How tall was the bamboo plant when it was planted?

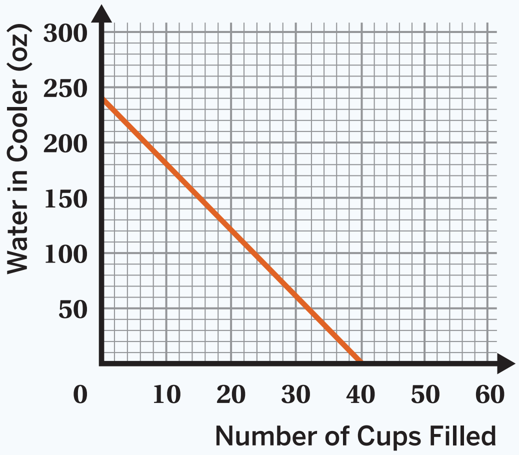
c Is this a linear relationship? Explain your thinking.



Summary | Lesson 5

When a *linear relationship* has a negative slope, this means that as the x-values increase, the y-values decrease at a constant rate.

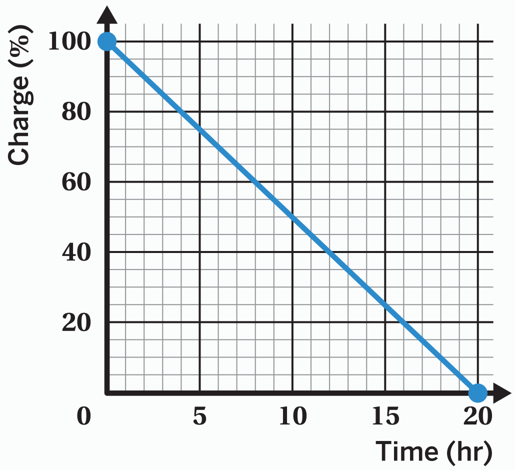
Let's say the equation $y = 240 - 6x$ represents the amount of water in a cooler, y, after x cups have been filled.



- The **vertical intercept**, also called the y-*intercept*, is $(0, 240)$. In this situation, the vertical intercept represents the starting amount of water in the cooler.
- The slope is -6. This means that the amount of water decreases by 6 ounces for each cup filled. Because the amount of water decreases each time, the slope is negative.
- The **horizontal intercept**, also called the x-*intercept*, is $(40, 0)$. In this situation, the horizontal intercept represents how many cups can be filled before the cooler runs out of water.

Try This

This graph shows a phone's charge over time.



a What is the vertical intercept of the line, and what does it represent?

b What is the horizontal intercept of the line, and what does it represent?

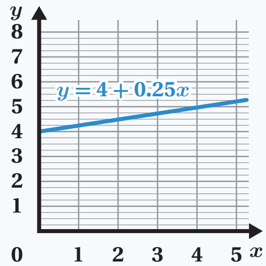
Equations of linear relationships, or linear equations, can be written in the form $y = mx + b$, where m represents the slope and b represents the vertical intercept.

For linear relationships with a *positive* slope, the y-values increase at a constant rate as the x-values increase. For linear relationships with a *negative* slope, the y-values decrease at a constant rate as the x-values increase.

Here are two examples.

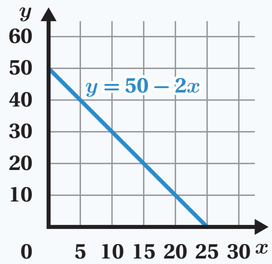
Positive slope: A medium-sized frozen yogurt costs $4, plus $0.25 per topping.

Let y represent the total cost of the frozen yogurt after adding x toppings.



Negative slope: A student loads an arcade game card with $50. Every time she plays a game, $2 is subtracted from the amount available on the game card.

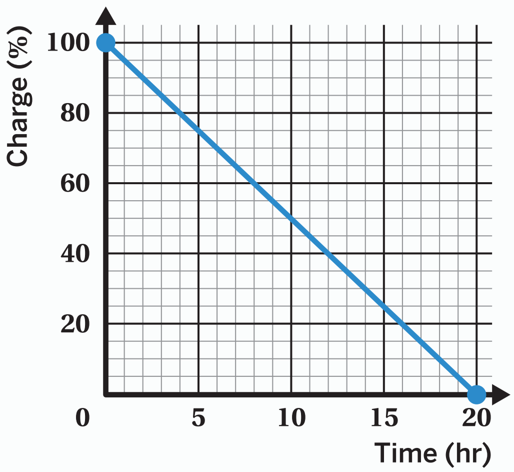
Let y represent the amount in dollars on the card after the student plays x games.



Try This

This graph shows a phone's charge over time.

Let y represent the percent charge after x hours.



a What is the slope of the line?

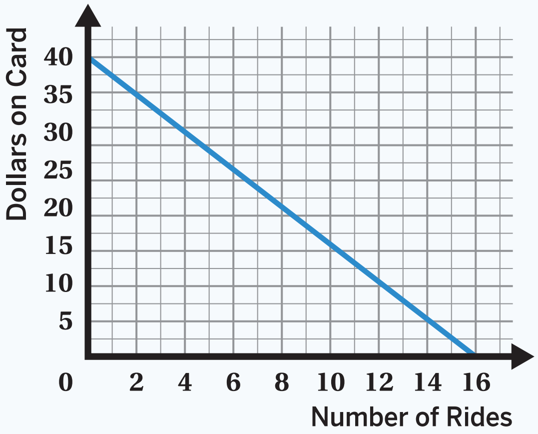
b What is the vertical intercept?

c Write an equation that represents this situation.

Summary | Lesson 7

Linear relationships can help us make predictions. We can use the graph or the equation of a linear relationship to determine the value of one variable when we're given the other variable.

Let's say we want to know how much money will be left on a transit card after 10 rides. We can look at a graph like this one and determine the amount of money on the card, y, that corresponds to $x = 10$. In this situation, the slope of the line tells us how much each ride costs.



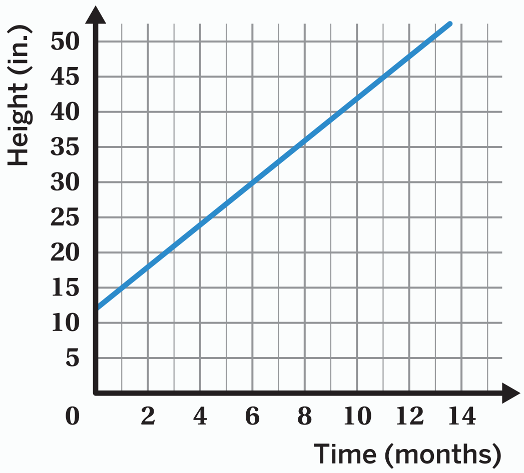
The y-intercept tells us the amount of money on the card before taking any rides. Unlike a proportional relationship, the graph of this linear relationship doesn't pass through the origin.

Try This

This graph shows the height, in inches, of a bamboo plant each month after it was planted.

The equation of this line is $y = 3x + 12$.

How tall will the bamboo plant be after 12 months?



Summary | Lesson 8

A *translation* of a line that represents a proportional relationship creates a line that is parallel to the pre-image, but changes the location of the vertical intercept, also known as the y-intercept.

The equation $y = mx$ represents a line that passes through the origin. The equation $y = mx + b$ represents a vertical translation of line $y = mx$ by b units.

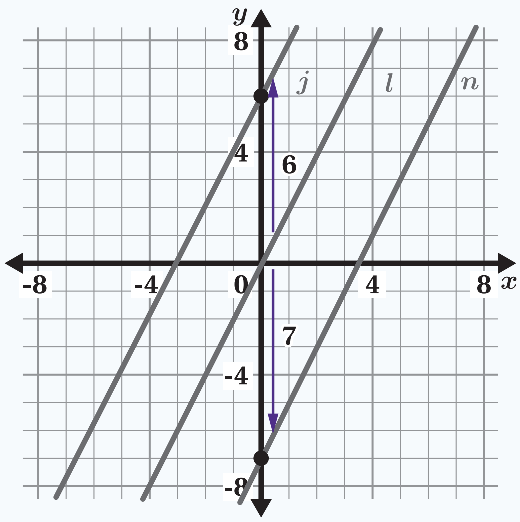
If $b > 0$, the line is translated up.

If $b < 0$, the line is translated down.

For example, the equation of line l is $y = 2x$.

If line l is translated 6 units up to produce line j, the equation of line j is $y = 2x + 6$.

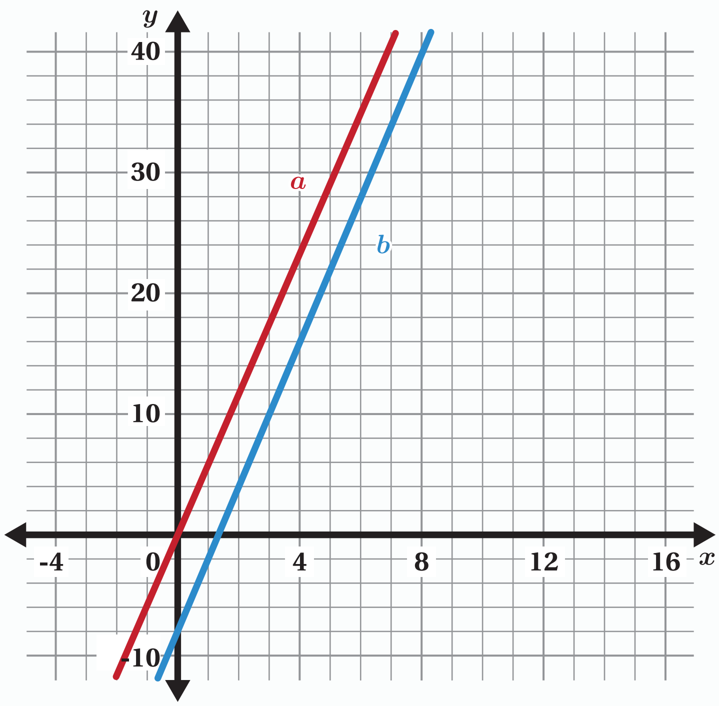
If line l is translated 7 units down to produce line n, the equation of line n is $y = 2x - 7$.



Try This

Here are the graphs of line a and line b. The equation for line a is $y = 6x$.

Write the equation for line b.



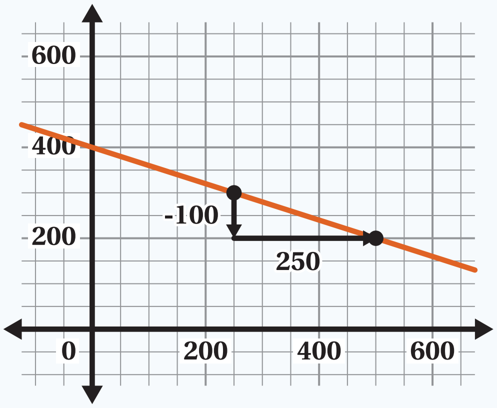
Summary | Lesson 9

You can determine the slope of a line using two points on that line. Lines with positive slopes increase in height from left to right, while lines with negative slopes decrease in height from left to right.

You can use slope triangles to calculate the vertical change and horizontal change between two points on a coordinate plane. You can also calculate the slope by listing the coordinates in a table and then determining the difference between the y-coordinates (the vertical change) and the difference between the x-coordinates (the horizontal change).

The slope is the ratio of the vertical change to the horizontal change.

Using Slope Triangles



Using Coordinates in a Table

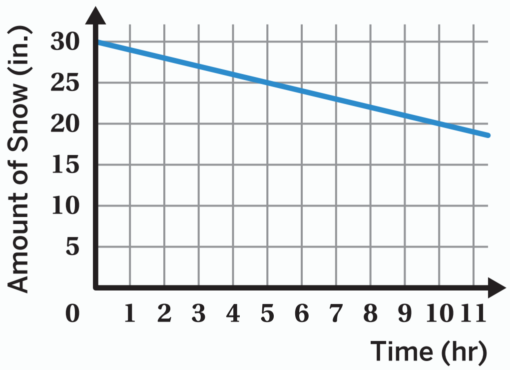
	x	y	
+250	250	300	-100
	500	200	

$$\frac{\text{change in } y \text{ (vertical change)}}{\text{change in } x \text{ (horizontal change)}} = \frac{-100}{250} = -\frac{2}{5}$$

Try This

This graph represents how much snow there is on the ground after it starts to melt in the sun.

What is the slope of the line?



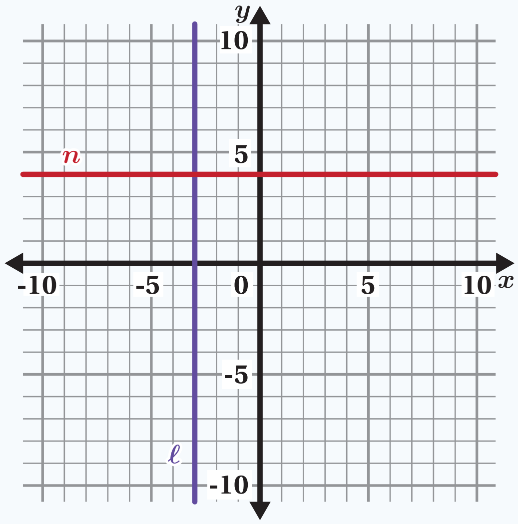
Summary | Lesson 10

On the coordinate plane:

- Horizontal lines represent situations where the y-value is constant and the x-values change. Horizontal lines have a slope of 0.
- Vertical lines represent situations where the x-value is constant and the y-values change. Vertical lines have an *undefined* slope.

For example, the equation $y = 4$ represents the horizontal line n because every point on the line has the same y-coordinate, 4.

The equation $x =$ -3 represents the vertical line ℓ because every point on the line has the same x-coordinate, -3.

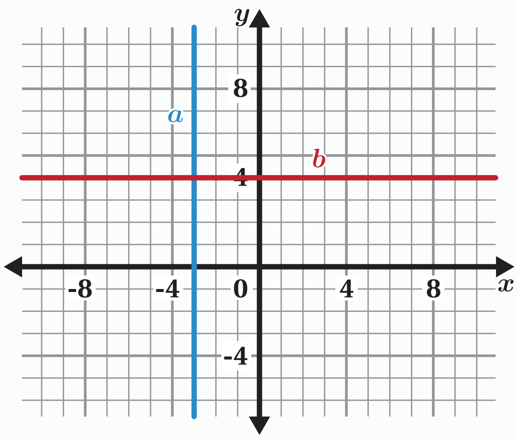


Try This

Write an equation for each line:

a Line a: ____________________

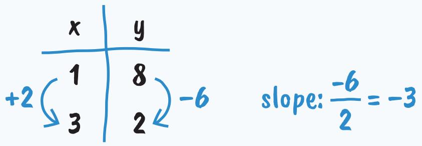
b Line b: ____________________



Here are two different strategies for writing the equation of a line using two given points.

Strategy Using a Table

First, calculate the slope using a table. Next, substitute the coordinates of one of the points into the equation $y = mx + b$ to determine the y-intercept. Then write the equation in the form $y = mx + b$.



$y = -3x + b$

Substitute (1, 8) in for x and y.

$8 = -3(1) + b$

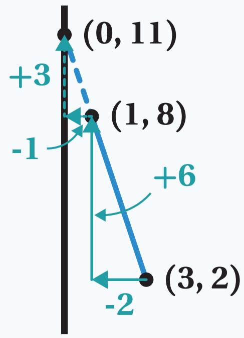
$8 = -3 + b$

$11 = b$

$y = -3x + 11$

Strategy Using Slope Triangles

Draw a line and use similar triangles to determine the slope and y-intercept of the line. Then write the equation in the form $y = mx + b$.



$8 + 3 = 11$

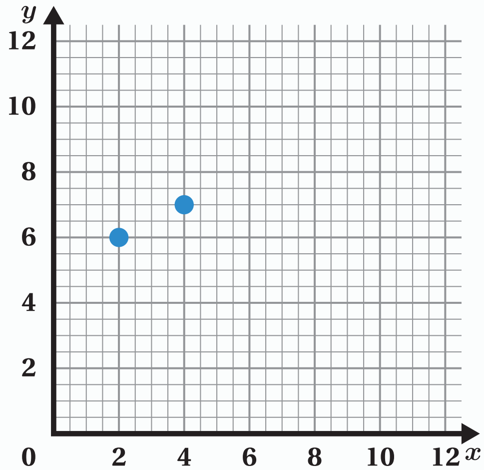
y-intercept: $(0, 11)$

Slope: -3

$y = -3x + 11$

Try This

Write an equation of a line that goes through points $(2, 6)$ and $(4, 7)$.

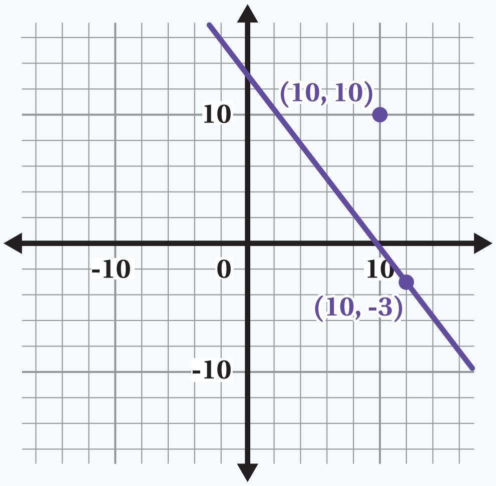


A **solution** to an equation with two variables is a set of values that makes the equation true. Solutions are often written as an ordered pair, (x, y).

Every point that lies on a line is a solution to that equation. Points that do not lie on the line are *not* solutions to the equation.

Here is a graph of the linear equation $3x + 2y = 24$.

- $(10, -3)$ is a solution to the equation $3x + 2y = 24$ because the point is on the graph of the line, and because $3(10) + 2(-3) = 24$.
- $(10, 10)$ is *not* a solution because the point is not on the line, and because $3(10) + 2(10) = 50$, not 24.
- Although we can't see it on the graph, $(-10, 27)$ is also a solution because $3(-10) + 2(27) = 24$.

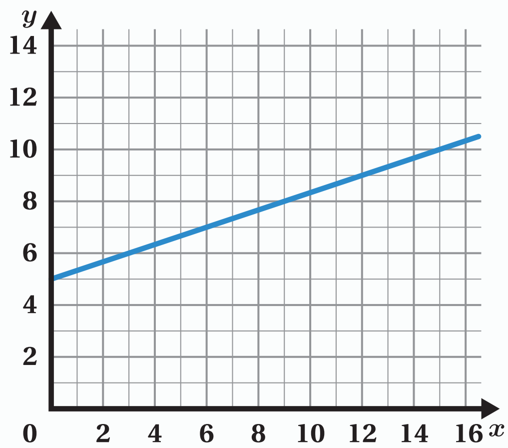


Try This

This graph shows the line $y = \frac{1}{3}x + 5$.

a Is $(10, 8)$ a solution to this equation? Explain your thinking.

b Is $(6, 7)$ a solution to this equation? Explain your thinking.



The four representations of a linear relationship — table, graph, equation, and verbal description — are all useful when solving real-world problems.

Let's say a coach has a $\$120$ budget to buy dinner for their team. Pizzas cost $\$20$ and sandwiches cost $\$8$. x represents the number of pizzas bought and y represents the number of sandwiches bought.

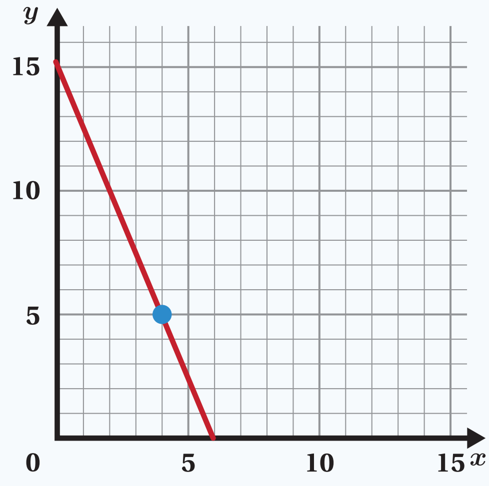
This situation can be modeled by the linear relationship $20x + 8y = 120$.

Here are two ways to show that 4 pizzas and 5 sandwiches is one solution to the equation:

- The values $x = 4$ and $y = 5$ make the equation true.

$$20(4) + 8(5) = 120$$
$$80 + 40 = 120$$
$$120 = 120$$

- The point $(4, 5)$ is on the graph of the linear relationship.



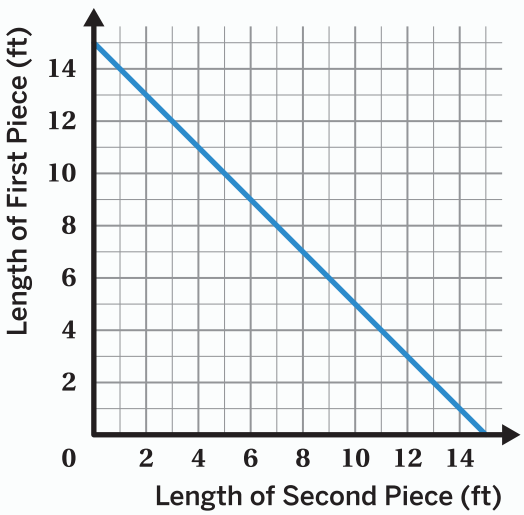
Try This

A 15-foot piece of ribbon is cut into two pieces.

This graph shows the possible lengths of each piece. The equation of the line is $y = -x + 15$.

a Is $(6, 9)$ a solution to this equation? Explain your thinking.

b Choose *one* solution to the equation and interpret what it means in this situation.



Lesson 1

a Yes. *Explanations vary*. Both lines pass through the origin, $(0, 0)$, so they both represent proportional relationships.

b Jasmine: $\frac{1}{4}$ or 0.25 miles per minute. Sothy: $\frac{1}{3}$ or about 0.33 miles per minute. Caregiver Note: To find Jasmine's speed, divide 4 miles by 16 minutes, or $\frac{4}{16} = \frac{1}{4}$. To find Sothy's speed, divide 4 miles by 12 minutes, or $\frac{4}{12} = \frac{1}{3}$.

Lesson 2

a Jasmine: $\frac{1}{4}$ or 0.25. Sothy: $\frac{1}{3}$ or about 0.33.

Caregiver Note: To find the slope of Jasmine's line, divide 4 miles by 16 minutes, or $\frac{4}{16} = \frac{1}{4}$. To find the slope of Sothy's line, divide 4 miles by 12 minutes, or $\frac{4}{12} = \frac{1}{3}$.

b Jasmine: $y = \frac{1}{4}x$

Sothy: $y = \frac{1}{3}x$

Lesson 3

Table

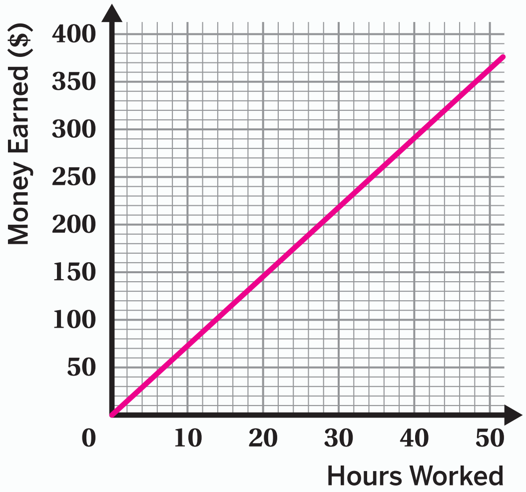
Responses vary.

Hours Worked	Money Earned ($)
10	72.50
20	145
30	217.50
40	290

Equation

$y = 7.25x$

Graph



Lesson 4

a 3. *Responses vary*. This means the bamboo plant grows 3 inches each month.

b 12 inches.

Caregiver Note: The graph starts at $(0, 12)$, which means that the plant was 12 inches tall when it was planted, at 0 months.

c Yes. *Explanations vary*. The graph forms a straight line.

Lesson 5

a. (0, 100). *Responses vary.* The phone's charge was 100% at 0 hours.

b. (20, 0). *Responses vary.* The phone's charge was at 0% after 20 hours.

Lesson 6

a. -5

b. (0, 100)

c. $y = -5x + 100$ (or equivalent)

Lesson 7

48 inches.

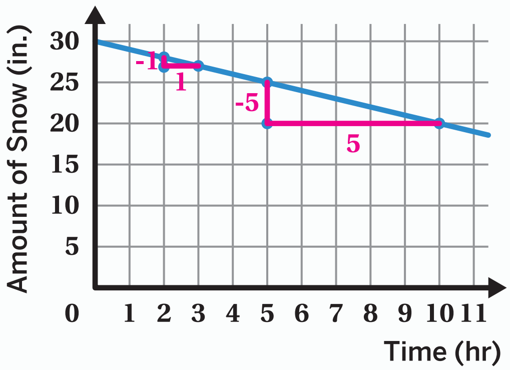
Caregiver Note: $y = 3(12) + 12 = 48$

Lesson 8

$y = 6x - 8$

Lesson 9

-1. *Caregiver Note: Here is one strategy for determining the slope using slope triangles:*



Slope: $\frac{-1}{1} = -1$ or $\frac{-5}{5} = -1$

Lesson 10

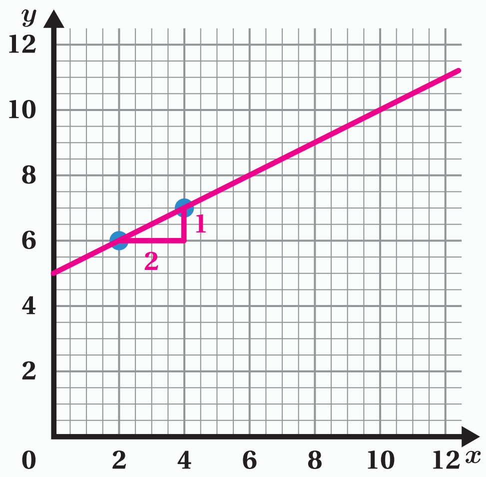
a. $x = -3$

b. $y = 4$

Lesson 11

$y = \frac{1}{2}x + 5$.

Caregiver Note: Here is one strategy for determining the equation using a slope triangle:



The slope triangle shows that the change in y-values is 1 and the change in x-values is 2, which means the slope is $\frac{1}{2}$. The y-intercept is $(0, 5)$, so the equation is $y = \frac{1}{2}x + 5$.

Lesson 12

a No. *Explanations vary.* The point $(10, 8)$ is not on the line and $8 \neq \frac{1}{3}(10) + 5$.

b Yes. *Explanations vary.* The point $(6, 7)$ is on the line and $7 = \frac{1}{3}(6) + 5$.

Lesson 13

a Yes. *Explanations vary.* The point $(6, 9)$ is on the line and $9 = \text{-}6 + 15$. This makes sense because if a 15-foot ribbon is cut so that one piece is 6 feet long, the other piece would be 9 feet long.

b *Responses vary.* The point $(10, 5)$ means that if the first piece is 5 feet long the second piece is 10 feet long.