Unit 4

Linear Equations and Linear Systems

Equations can help you understand and solve problems. So far, you've solved single variable equations where the variable is on one side. In this unit, you'll solve equations with variables on both sides. You'll also solve two linear equations in a system and determine how many solutions there are.

Essential Questions

- How can you solve an equation with variables on both sides of the equal sign?

- How can systems of equations be used to represent situations and solve problems?

- What does it mean for an equation or system of equations to have *no*, *one*, or *infinitely many* solutions?

You can use a number machine to perform a series of operations on a number.

- If only the *input* (the number put into the machine) is provided, the *output* (the number that comes out of the machine) can be determined by performing the series of operations in order.
- If only the output is provided, the input can be determined by guessing and checking, by working backward, or by writing and solving an equation.

For example, a number machine takes an input and performs these steps in order:

- Multiply by 3
- Add 8
- Subtract 1

The output is 19. Here are two ways you can determine the input:

Work backward from the output	**Write and solve an equation**
Start with 19, add 1, subtract 8, and then divide by 3. $(19 + 1 - 8) \div 3 = 4$.	The equation $(3x + 8) - 1 = 19$ represents the number machine, where x is the input.
These steps show that the input is 4.	Solving this equation shows that $x = 4$.

Try This

A number machine takes an input and performs these operations in order:
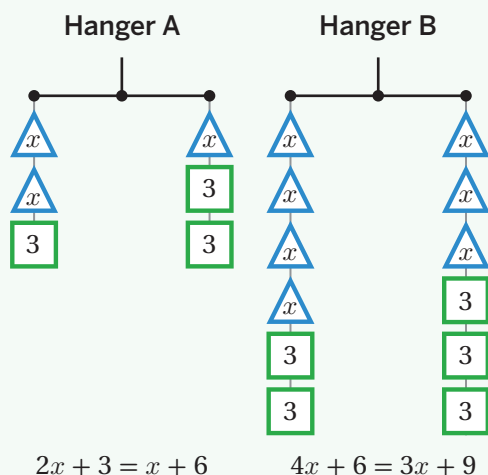
- Multiply by 2.
- Add 4.

The output is 18. What was the input?

Summary | Lesson 2

A hanger diagram is balanced when its left and right sides are equal in weight. Adding or removing equally weighted items from each side of a hanger diagram keeps the hanger balanced. Balanced hanger diagrams can represent equations.

For example, Hangers A and B represent **equivalent equations**, which are equations that have the same solution.

These equations are equivalent because if you add $2x$ and 3 to each side of Hanger A, you create the also balanced Hanger B.

The solution to each equation is $x = 3$, which is the weight of each triangle on both hangers.
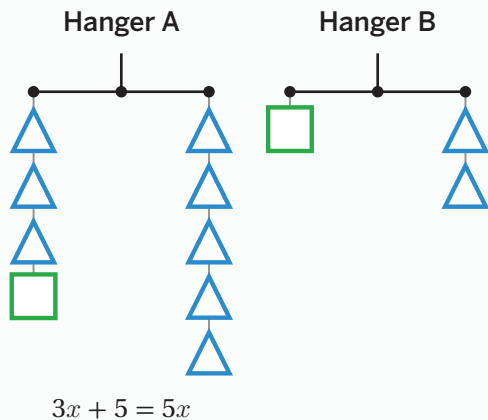
Hanger A

Hanger B

$$2x + 3 = x + 6 \qquad 4x + 6 = 3x + 9$$

Try This

Hangers A and B are both balanced hangers. Each square weighs 5 pounds.

Hanger A can be represented by the equation $3x + 5 = 5x$.

a Write an equivalent equation to represent Hanger B.

b Determine the weight of the triangle.

Hanger A

Hanger B

$$3x + 5 = 5x$$

You can use hanger diagrams to visualize and represent equations. Hanger diagrams can help you see how moves that keep the hanger balanced create *equivalent equations*, such as:

- Adding or subtracting the same term on each side of an equation.
- Multiplying or dividing the expressions on each side of an equation by the same number.

You can use these moves to solve equations for an unknown variable. Here is an example of using balanced moves to solve an equation:

Solving Step	Balanced Equation Move
$\begin{aligned} 4x + 9 &= \text{-}2x - 3 \\ -\ 9 &\qquad -\ 9 \end{aligned}$	Subtract 9 from both sides.
$\begin{aligned} 4x &= \text{-}2x - 12 \\ +\ 2x &+ 2x \end{aligned}$	Add $2x$ to both sides.
$\dfrac{6x}{6} = \dfrac{\text{-}12}{6}$	Divide by 6 on both sides.
$x = \text{-}2$	

You can check if a value is a *solution* to an equation by substituting it into the original equation. If it makes the equation true then the solution is correct.

$$4(\text{-}2) + 9 = \text{-}2(\text{-}2) - 3$$
$$\text{-}8 + 9 = 4 - 3$$
$$1 = 1$$

Try This

a Here are the steps to solve an equation. Match each step with a description of how it balances the equation.

Solving Steps

Step 1:
$4x + 9 = \text{-}2x - 3$
$\qquad 4x = \text{-}2x - 12$

Step 2:
$4x = \text{-}2x - 12$
$6x = \text{-}12$

Step 3:
$6x = \text{-}12$
$\qquad x = \text{-}2$

Descriptions

Step _____: Add $2x$ to both sides.

Step _____: Subtract 9 from both sides.

Step _____: Divide both sides by 6.

b How can you check if -2 is a solution to the equation?

There are many ways to solve an equation using balanced equation moves. Generally, you want to perform steps that will move you closer to an equivalent equation where the variable is isolated, such as $x = 5$, or $10 = y$. Here are some examples of moves that can be helpful in solving an equation.

Applying the *distributive property*	Combining *like* terms on one side of an equation	Adding or subtracting the same term from both sides	Multiplying or dividing both sides of the equation by the same value
$2x + 4 = 3(2x + 4)$ $2x + 4 = 6x + 12$	$\text{-}10x + 3x = 15$ $\text{-}7x = 15$	$\text{-}x - 8 = 4x + 7$ $\text{-}8 = 5x + 7$	$\text{-}15x = 5(2x + 4)$ $\text{-}3x = 2x + 4$

Try This

Solve the equation $2x = \text{-}3(x + 5)$. Show your thinking.

You can use different steps to solve the same equation. For example, here are two ways to solve the equation $\frac{1}{3}(3x + 9) = 6x + 18$:

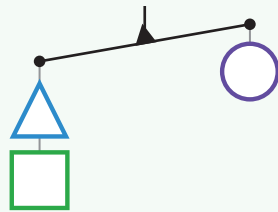| Start by distributing $\frac{1}{3}$ to $(3x + 9)$	Start by multiplying both sides of the equation by 3
$\frac{1}{3}(3x + 9) = 6x + 18$ $x + 3 = 6x + 18$ $-5x + 3 = 18$ $-5x = 15$ $x = -3$	$\frac{1}{3}(3x + 9) = 6x + 18$ $3x + 9 = 18x + 54$ $3x - 45 = 18x$ $-45 = 15x$ $-3 = x$

Sometimes you might unintentionally make a move that unbalances an equation. Here are some examples:

	Unbalanced Move	Balanced Move
Distributing a factor to some terms in parentheses but not all of them	$\frac{1}{3}(3x + 9) = 5x$ $x + 9 \neq 5x$	$\frac{1}{3}(3x + 9) = 5x$ $x + 3 = 5x$
Multiplying some terms in an equation by a factor but not all of them	$\frac{1}{5}(x + 2) = 4x + 6$ $x + 2 \neq 20x + 6$	$\frac{1}{5}(x + 2) = 4x + 6$ $x + 2 = 20x + 30$
Adding or subtracting a term instead of distributing	$7 - 4(x + 1) = 2x + 5$ $3(x + 1) \neq 2x + 5$	$7 - 4(x + 1) = 2x + 5$ $7 - 4x + 4 = 2x + 5$

Try This

a What is one possible first step you could take to solve this equation?
$9 - 2b + 6 = -3(b + 5) + 4b$

b What is a different possible first step you could take to solve this equation?
$9 - 2b + 6 = -3(b + 5) + 4b$

c Does this move keep the equation balanced? Explain your thinking.
$9 - 2b + 6 = -3(b + 5) + 4b$
$9 + 6 = -3(b + 5) + 6b$

Equations can have one solution, no solution, or **infinitely many solutions**.

Here are some examples.

One Solution	No Solution	Infinitely Many Solutions
$3x + 8 = 6 + 2 - 3x$	$3(x + 4) = 3x + 7$	$10 - 3x = 8 - 3x + 2$
This equation is only true when $x = 0$.	This equation is never true for any value of x.	This equation is always true for any value of x.
A linear equation has *one solution* when the expressions on either side of the equation have one value for the variable that makes them equal.	A linear equation has *no solution* when the expressions on either side of the equation have no value for the variable that make them equal.	A linear equation has *infinitely many solutions* when the expressions on either side of the equation are *equivalent*: always equal no matter the value of the variable.

Try This

Determine if each equation has one solution, no solution, or infinitely many solutions.

$2x = x + x$	$3x = x + 2$	$7 + x = x + 7$
$x = x + 1$	$-x = 1 - x$	$9x = 10$

One Solution	No Solution	Infinitely Many Solutions

Equations can have many different features, including fractions, decimals, negative values, grouping symbols, and multiple terms. Based on the features, it can be helpful to think about what steps might be most useful in solving the equation.

When solving an equation with one solution, the goal is to end up with the variable isolated on one side of the equation and its value on the other. But this doesn't happen when there is no solution or infinitely many solutions.

One Solution	No Solution	Infinitely Many Solutions
$3x + 8 = 6 + 2 - 3x$	$3(x + 4) = 3x + 7$	$10 - 3x = 8 - 3x + 2$
$3x + 8 = 8 - 3x$	$3x + 12 = 3x + 7$	$10 - 3x = 10 - 3x$
$6x + 8 = 8$	$12 = 7$	$10 = 10$
$6x = 0$		
$x = 0$		
This equation is only true when $x = 0$.	This equation is never true for any value of x.	This equation is always true for any value of x.

Try This

Solve the equation. Show your thinking.

$3x - 7x + 5 = 3(x - 10)$

We can write two expressions in one variable and set them equal to each other to represent a scenario in which two conditions are equal. We can solve this equation to determine the unknown quantity.

For example, imagine two hikers walking in the same direction on a flat trail. The hikers will meet each other when they are at the same location on the trail at the same time.

To determine when this occurs, an expression can be used to represent the location and walking speed of each hiker.

	Location (ft)	Walking Speed (ft/s)	Expression
Hiker 1	30	4	$30 + 4t$
Hiker 2	10	7	$10 + 7t$

You can set these two expressions equal to each other to form one equation that can be solved.

$$30 + 4t = 10 + 7t$$

$$20 = 3t$$

$$t = \tfrac{20}{3} \text{ or about 6.7 seconds}$$

Try This

A car is traveling at a constant speed of 16 meters per second. A scooter is traveling at 9 meters per second and is 42 meters ahead of the car.
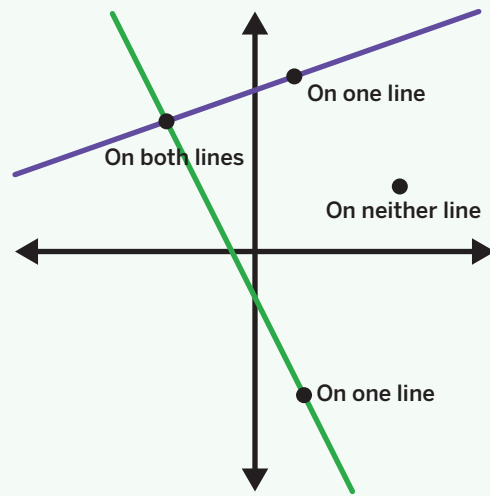
This situation can be represented by the equation $16t = 9t + 42$, where t represents the number of seconds that the vehicles have been traveling.

a Solve this equation. Show your thinking.

b What does the solution represent in this situation?

Linear relationships can represent many situations. Lines graphed on the same coordinate plane can simultaneously represent multiple conditions or relationships involving the same variables.

- The coordinates of a point that is on both lines make both relationships true.
- The coordinates of a point on only one line make only one relationship true.
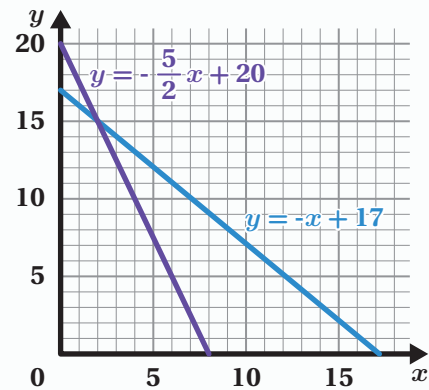- The coordinates of a point on neither line make neither relationship true.

Try This

This graph represents two relationships.

a What is a combination of values that makes both relationships true?

$x =$ _____ $y =$ _____

b What is a combination of values that makes *one* relationship true but not the other?

$x =$ _____ $y =$ _____

$$y = -\frac{5}{2}x + 20$$

$$y = \text{-}x + 17$$
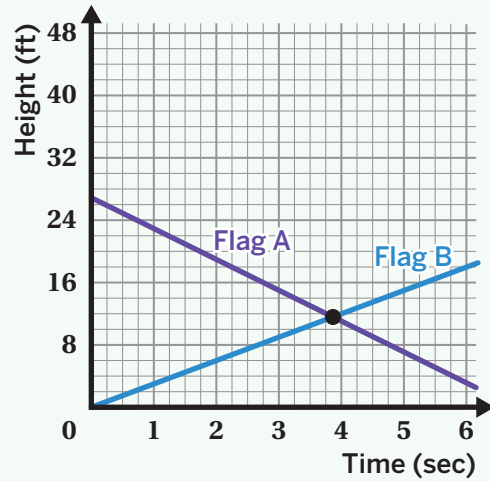
If there are two equations that share the same variables, you can find the solution that makes both equations true by locating the **point of intersection**, where the two lines meet on a graph.

For example, consider this graph.

Although you can't see the exact values of the point of intersection, you can tell that the flags are the same height, at about 11.5 feet, just before 4 seconds.
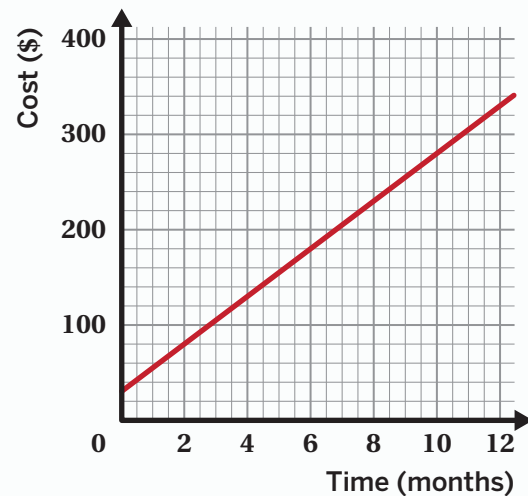

Try This

Gym A costs $30 to join and $25 per month. Gym B costs $100 to join and $15 per month.

The graph shows the cost of Gym A over time.

a Graph a line to represent the cost of Gym B over time.

b Explain what the point of intersection represents in this situation.
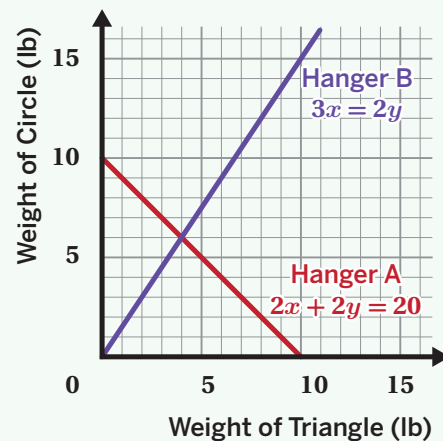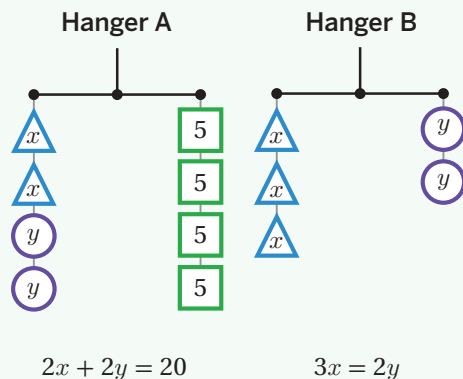
A **system of equations** is a set of two or more equations with the same variables that are meant to be solved together. It is sometimes shown with a single bracket around the equations.

A **solution to a system** of equations is a set of values that makes all equations in that system true.

For example, here is a system of equations:

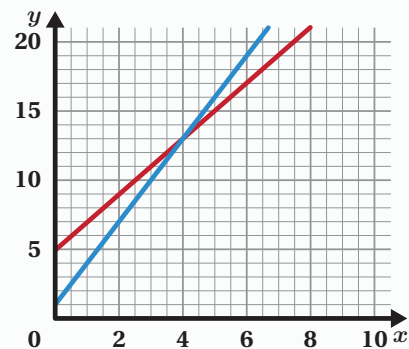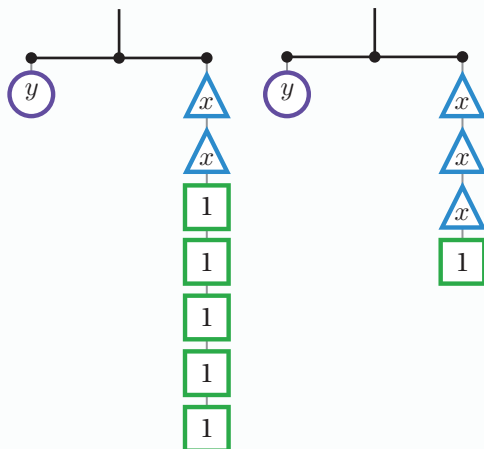$$\begin{cases} 2x + 2y = 20 \\ 3x = 2y \end{cases}$$

The ordered pair $(4, 6)$ is the point of intersection, which means that it will make both equations true. Both hangers will balance when the triangles weigh 4 pounds and the circles weigh 6 pounds.

Try This

Determine the solution to this system of equations:

$y = 2x + 5$

$y = 3x + 1$

For a point to be a solution to a system of equations, the x- and y-coordinates must make both of the equations true. This ordered pair is the *point of intersection* when the system is graphed.

For example, here is a system of equations:

$$\begin{cases} y = 4x - 5 \\ y = \text{-}2x + 7 \end{cases}$$

To determine the solution to the system, you can write a single equation by taking the two expressions that are equal to y and setting them equal to each other.

$4x - 5 = \text{-}2x + 7$
$6x - 5 = 7$
$6x = 12$
$x = 2$

You can then substitute the solution for x into either of the original equations to determine the value of y.

$y = 4x - 5$
$y = 4(2) - 5$
$y = 8 - 5$
$y = 3$

The solution to this system of equations is the point (2, 3).
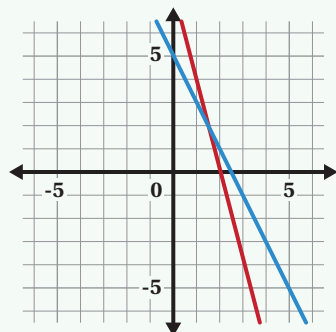
Try This

What is the solution to this system of equations?

$y = 2x + 6$

$y = \text{-}3x - 4$

Systems of two linear equations can have one solution, no solution, or infinitely many solutions. You can determine the number of solutions to a system of equations by graphing, comparing the slopes and y-intercepts, or solving the system algebraically.
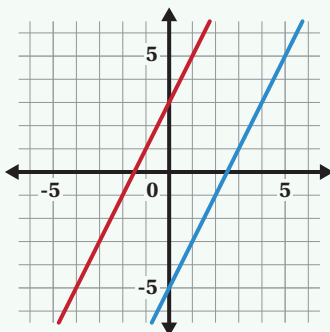
One solution:	No solution:	Infinitely many solutions:

$$\begin{cases} y = \text{-}4x + 8 \\ y = \text{-}2x + 5 \end{cases}$$

- Different slopes

$$\begin{cases} y = 2x + 3 \\ y = 2x - 5 \end{cases}$$

- Same slopes
- Different y-intercepts

$$\begin{cases} y = 2x + 3 \\ y = 2x + 3 \end{cases}$$
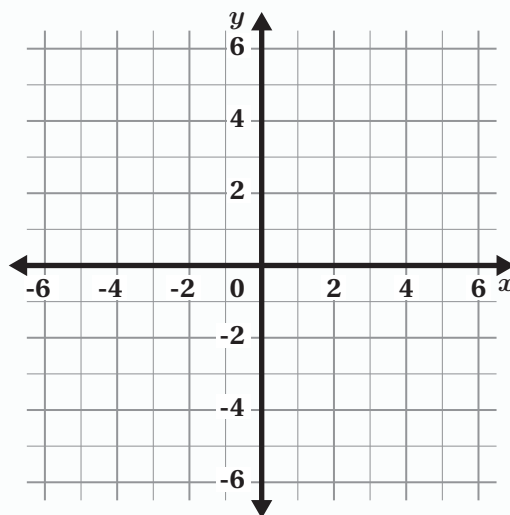
- Same slopes
- Same y-intercepts

Try This

Determine whether each system of equations will have one solution, no solution, or infinitely many solutions. Use the graph if it helps with your thinking. Show or explain your thinking.

a
$$\begin{cases} y = \frac{2}{3}x - 2 \\ y = \frac{2}{3}x + 3 \end{cases}$$

b
$$\begin{cases} y = 1.5x \text{-}2 \\ y = 1.5x - 2 \end{cases}$$

An ordered pair (x, y) is a solution to a system of equations if it makes both equations true. If you know the value of one variable in one of the equations, you can substitute it into the other equation to solve for the second variable.

Here is a system of equations:

$$\begin{cases} y = 5x \\ 2x - y = 9 \end{cases}$$

Since $y = 5x$, you can substitute $5x$ for y in $2x - y = 9$, and then solve for x.

You can then substitute the solution for x into either of the original equations to determine the value of y.

$2x - (5x) = 9$

$-3x = 9$

$x = -3$

$y = 5x$

$y = 5(-3)$

$y = -15$

The ordered pair (-3, -15) is the solution to the system of equations.

Try This

Solve this system of equations. Show your thinking.

$$\begin{cases} x = 7 \\ y = -3x + 42 \end{cases}$$

Try This | Answer Key

Lesson 1

7

Caregiver Note: Here are two strategies for calculating the answer:

Work backward from the output: Start with 18. Subtract 4, then divide by 2. $(18 - 4) \div 2 = 7$

Write and solve an equation: The equation $2x + 4 = 18$ represents the number machine, where x is the input. Solving this equation shows that $x = 7$.

Lesson 2

a $5 = 2x$

b 2.5 pounds

Lesson 3

a Step __**2**__: Add $2x$ to both sides.

Step __**1**__: Subtract 9 from both sides.

Step __**3**__: Divide both sides by 6.

b You can check if a value is a solution to an equation by substituting it into the original equation.
$4x + 9 = \text{-}2x - 3$
$4(\text{-}2) + 9 = \text{-}2(\text{-}2) - 3$
$(\text{-}8) + 9 = (4) - 3$
$1 = 1$

Lesson 4

$x = \text{-}3$. Work varies.

$2x = \text{-}3(x + 5)$

$2x = \text{-}3x - 15$

$5x = \text{-}15$

$x = \text{-}3$

Lesson 5

a *Responses vary.* One possible first step is to distribute -3 to $(b + 5)$:
$$9 - 2b + 6 = \text{-}3(b + 5) + 4b$$
$$9 - 2b + 6 = \text{-}3b - 15 + 4b$$

b *Responses vary.* Another possible first step is to combine like terms 9 and 6:
$$9 - 2b + 6 = \text{-}3(b + 5) + 4b$$
$$15 - 2b = \text{-}3(b + 5) + 4b$$

c Yes. *Explanations vary.* $2b$ was added to both sides of the equation, so the equation remains balanced.

Lesson 6

One Solution	No Solution	Infinitely Many Solutions
$3x = x + 2$	$x = x + 1$	$7 + x = x + 7$
$9x = 10$	$\text{-}x = 1 - x$	$2x = x + x$

Lesson 7

$x = 5$. *Work varies.*

$$3x - 7x + 5 = 3(x - 10)$$
$$\text{-}4x + 5 = 3(x - 10)$$
$$\text{-}4x + 5 = 3x - 30$$
$$5 = 7x - 30$$
$$35 = 7x$$
$$5 = x$$

Lesson 8

a $t = 6$. *Work varies.*

$$16t = 9t + 42$$
$$7t = 42$$
$$t = 6$$

b *Responses vary.* The car will catch up to the scooter after 6 seconds.

Try This | Answer Key
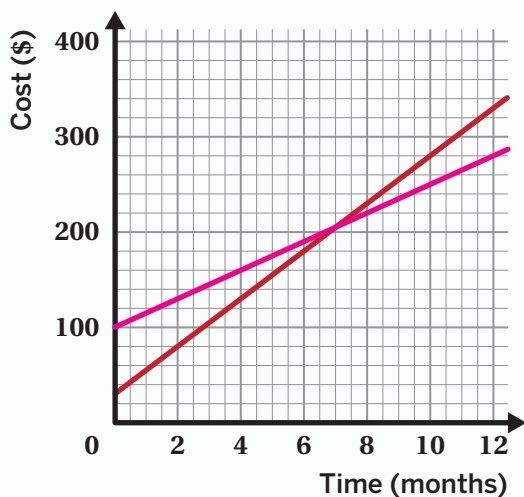
Lesson 9

a $x = 2$ and $y = 15$

Caregiver Note: One strategy is to look for a point that is on both lines. $(2, 15)$ is that point.

b *Responses vary.* $x = 6$ and $y = 5$.

Caregiver Note: One strategy is to look for a point that is on only one line, such as $(6, 5)$.

Lesson 10

a

b The point of intersection represents the month during which both gyms will cost an equal amount. After 7 months both gyms will cost a bit more than $200.

Lesson 11

(4, 13)

Caregiver Note: One strategy is to look at the point of intersection for the lines, which represents the x- and y-values that make both equations true.

Lesson 12

(-2, 2)

Caregiver Note: One strategy is to write a single equation by taking the two expressions that are equal to y and setting them equal to each other.

$2x + 6 = \text{-}3x - 4$

$5x + 6 = \text{-}4$

$5x = \text{-}10$

$x = \text{-}2$

Then substitute the solution for x into one of the original equations:

$y = 2x + 6$

$y = 2(\text{-}2) + 6$

$y = \text{-}4 + 6$

$y = 2$

So the solution to this system of equations is (-2, 2).

Lesson 13

a No solution. Since these lines have the same slope and different y-intercepts, they will never intersect.

b Infinitely many solutions. Since these are the same line, any point on one line will be on both lines.

Lesson 14

(7, 21). *Work varies.* Here is one strategy:

Since $x = 7$, you can substitute 7 for x in $y = \text{-}3x + 42$:

$y = \text{-}3x + 42$

$y = \text{-}3(7) + 42$

$y = \text{-}21 + 42$

$y = 21$

The ordered pair (7, 21) is the solution to the system of equations.